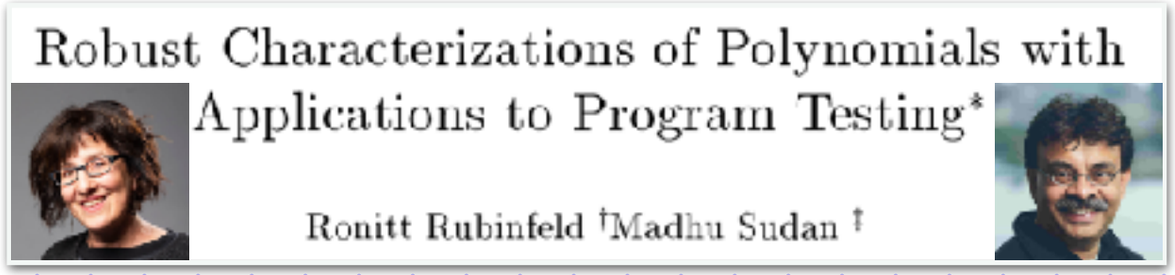
Foundations of Probabilistic Proofs

A course by **Alessandro Chiesa**

Lecture 09

Low-Degree Testing

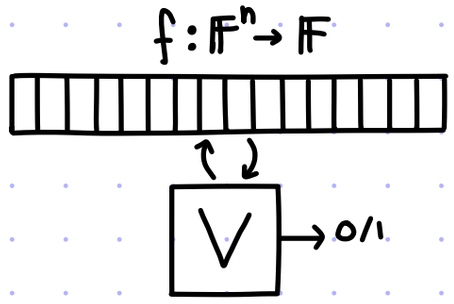
Low-Degree Testing

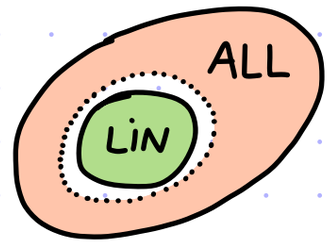


Recall the goal of LINEARITY TESTING:

A test V_{LIN} s.t. $\forall f: \mathbb{F}^n \to \mathbb{F}$

- completeness: $f \in LIN[\mathbb{F},n] \to Pr[V_{LIN}^f = 1] = 1$
- soundness: $\Delta(f, LIN[\mathbb{F},n]) \geq \delta \to Pr[V_{LIN}^f = 1] \leq \varepsilon(\delta)$

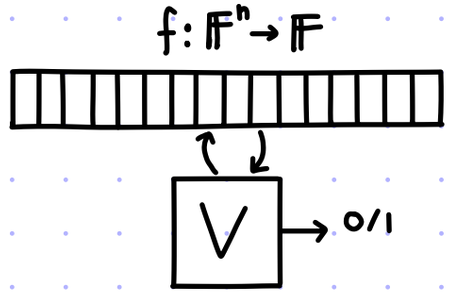


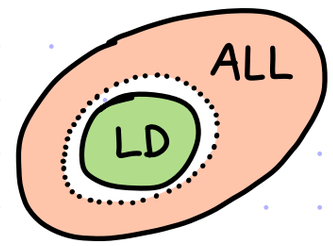


The goal of LOW-DEGREE TESTING is:

A test V_{LD} s.t. $\forall f: \mathbb{F}^n \to \mathbb{F}$

- completeness: $f \in LD[\mathbb{F},n,d] \to Pr[V_{LD}^f = 1] = 1$
- soundness: $\Delta(f, LD[\mathbb{F},n,d]) \geq \delta \to Pr[V_{LD}^f = 1] \leq \varepsilon(\delta)$





What does degree d mean?

- total degree: $LD[\mathbb{F},n,tot \leq d]$ (e.g. in this case $LD[\mathbb{F},n,tot \leq 1] = AFFINE[\mathbb{F},n]$)
- individual degree: $LD[\mathbb{F},n,ind \leq d]$ (e.g. in this case $LD[\mathbb{F},n,ind \leq 1] =$ "multilinear polynomials")

EXERCISE: derive a test for individual degree from a test for total degree.

In most applications the difference total-vs-individual does not matter much.

The Case of Low Total Degree

Today we study low-degree testing for the case of TOTAL degree:

Individual degree is $<|\mathbb{F}|$ since $\forall i \in [n].\ X_i^{|\mathbb{F}|} \equiv X_i$.

$$LD[\mathbb{F}, n, \text{tot} \le d] = \left\{ f : \mathbb{F}^n \to \mathbb{F} \;\middle|\; \exists\, p \in \mathbb{F}[X_1,\ldots,X_n] \text{ of TOTAL degree } \le d \text{ s.t. } p \equiv f \right\}.$$

This set of functions is known as the Reed-Muller code (RM code).

The RM code is a linear (error-correcting) code:

$LD[\mathbb{F}, n, \text{tot} \le d]$ is an $\mathbb{F}$-linear subspace of $\mathbb{F}^n$ $\left(\forall f,g \in LD[\mathbb{F},n,\text{tot}\le d]\ \forall \alpha,\beta \in \mathbb{F},\ \alpha f + \beta g \in LD[\mathbb{F},n,\text{tot}\le d]\right)$.

Typically $\mathbb{F}$ is large enough for soundness reasons.

The code's parameters depend on the regime. In this course we consider $d < |\mathbb{F}|$:

- message length $= \binom{n+d}{d}$. Size of the set $\{(d_1,\ldots,d_n) \in \{0,1,\ldots,|\mathbb{F}|-1\} \mid \sum_{i=1}^n d_i \le d\}$.
- block length $= |\mathbb{F}|^n$. Size of a function $f : \mathbb{F}^n \to \mathbb{F}$.
- relative distance $\ge 1 - \frac{d}{|\mathbb{F}|}$. By the Polynomial Identity Lemma $\left(\forall p \not\equiv 0 \ \Pr_{\alpha \leftarrow \mathbb{F}^n}[p(\alpha)=0] \le \frac{\deg_{\text{tot}}(p)}{|\mathbb{F}|}\right)$.

The other regime is $d \ge |\mathbb{F}|$ (e.g. RM code over $\mathbb{F}_2$).

In general (any d):

- message length $\le \min\left\{|\mathbb{F}|^n, \binom{n+d}{d}\right\}$. (Can give better bounds depending on $|\mathbb{F}|, n, d$.)
- relative distance $\ge \frac{|\mathbb{F}|-d_Q}{|\mathbb{F}|^{d_R+1}}$ where $d_Q \cdot (|\mathbb{F}|-1) + d_R = d$ for $d_R < |\mathbb{F}|-1$.

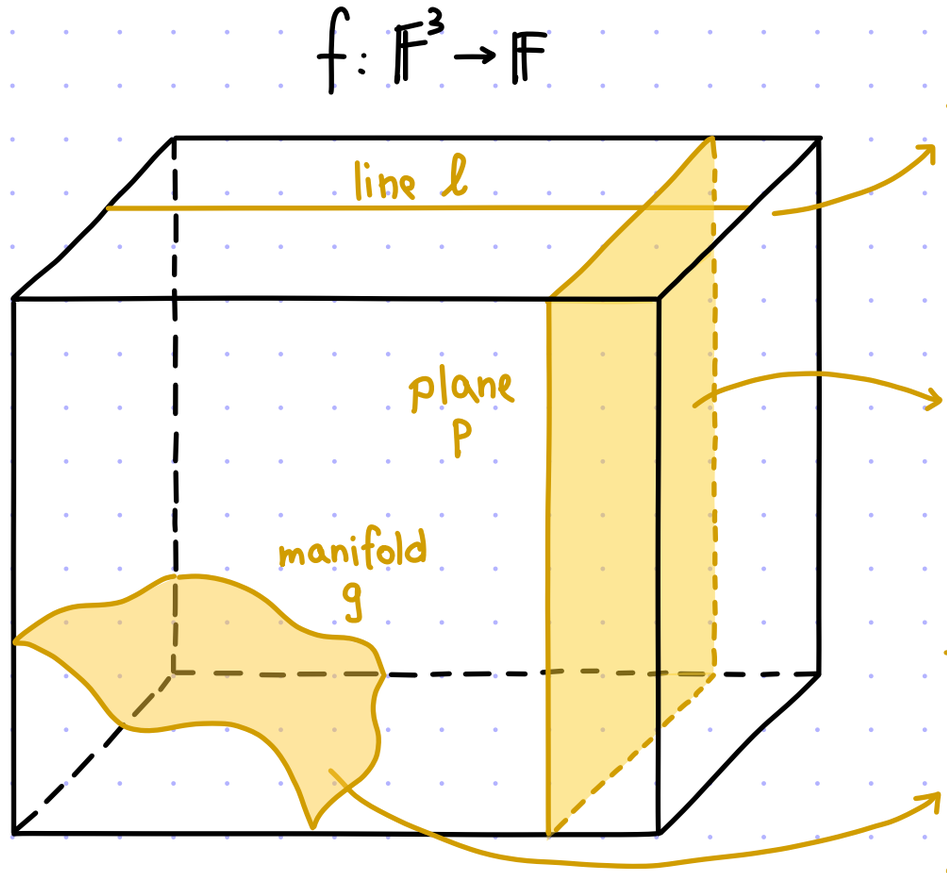
↖ implies bound for $d < |\mathbb{F}|$

The Rich Structure of the RM Code

The RM code LD[$\mathbb{F}$, n, tot$\leq$d] has a rich structure:

any low-degree (e.g. linear) restriction of the domain $\mathbb{F}^n$ yields a good subcode;

there are many such restrictions, and typically intersect with one another.



A line is $\ell: \mathbb{F} \to \mathbb{F}^3$ of total degree 1.

Hence if $\ell(z) = (a_0 + za_1, b_0 + zb_1, c_0 + zc_1)$

then $(f \circ \ell)(z) = f(a_0 + za_1, b_0 + zb_1, c_0 + zc_1)$.

A plane is $p: \mathbb{F}^2 \to \mathbb{F}^3$ of total degree 1.

Hence if $p(y,z) = (a_0 + a_1 y + a_2 z, b_0 + b_1 y + b_2 z, c_0 + c_1 y + c_2 z)$

then $(f \circ p)(y,z) = f(a_0 + a_1 y + a_2 z, b_0 + b_1 y + b_2 z, c_0 + c_1 y + c_2 z)$.

A k-variate manifold in $\mathbb{F}^3$ is $g: \mathbb{F}^k \to \mathbb{F}^3$.

Hence if $g(z_1, \ldots, z_k) = (g_1(z_1, \ldots, z_k), g_2(z_1, \ldots, z_k), g_3(z_1, \ldots, z_k))$

then $(f \circ g)(z_1, \ldots, z_k) = f(g_1(z_1, \ldots, z_k), g_2(z_1, \ldots, z_k), g_3(z_1, \ldots, z_k))$.

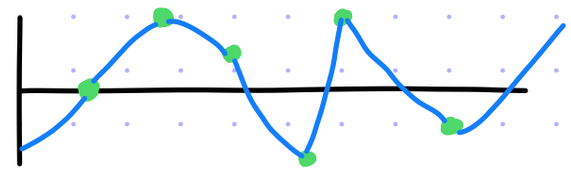
[A curve is a 1-variate manifold.]

We will see how this rich structure enables proximity testing.

Next: ① low-degree testing for LD[$\mathbb{F}$, n=1, d] (univariate polynomials)

② extend to LD[$\mathbb{F}$, n>1, tot$\leq$d] (multivariate polynomials)

Univariate Polynomials: a Trivial Test



Fact: any $d+1$ locations $a_0, a_1, \ldots, a_d \in \mathbb{F}$ determine a polynomial

A natural idea is to interpolate and test at a random point:

$V^{f:\mathbb{F}\to\mathbb{F}}(\mathbb{F}, d) :=$
1. Sample $r \leftarrow \mathbb{F}$
2. Query f at $a_0, a_1, \ldots, a_d, r$
3. Let $\tilde{p}(x)$ be the interpolation of $\{(a_i, f(a_i))\}_{i=0}^{d}$
4. Check that $\tilde{p}(r) = f(r)$

query complexity:
$d+2 = O(d)$
[& non-adaptive]

Completeness: if $f \equiv p$ for a polynomial $p(x)$ of degree $\leq d$
then $\tilde{p} = p$ and so $\forall r \in \mathbb{F}$ $\tilde{p}(r) = p(r) = f(r)$

Soundness: $\Pr[V^f = 1] = \Pr_r[\tilde{p}(r) = f(r)] \leq 1 - \Delta(f, LD[\mathbb{F}, n=1, d])$

The query complexity $O(d)$ can be much less than $|\mathbb{F}|$ (reading all of f).

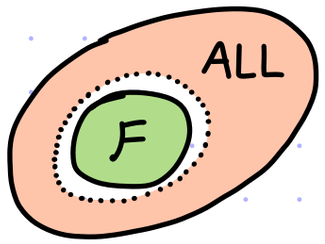
Exercise: prove that a query complexity of $\Omega(d)$ is necessary.

PROBLEM: the straightforward extension of this idea to $n>1$ yields large query complexity.

Trivial Test for Multivariate Polynomials

The "interpolate-and-test" idea is an example of a TRIVIAL TEST.

We describe a trivial test for any property $F = \{f : D \to \Sigma\}$.



A fixing set $S \subseteq D$ for F is such that, $\forall a : S \to \Sigma$, $|\{f \in F \mid f(S) = a\}| \leq 1$.

$V_S^{f:D\to\Sigma}$:
1. Sample $r \leftarrow D$.
2. Query f at S and r. ← query complexity is $|S|+1$
3. Let $\tilde{p}$ be the unique function in F s.t. $\tilde{p}(S) = f(S)$.
4. Check that $\tilde{p}(r) = f(r)$.

Completeness: if $f \in F$ then $\Pr[V_S^f = 1] = \Pr[\tilde{p}(r) = f(r)] = 1$.

Soundness: $\Pr[V_S^f = 1] = \Pr[\tilde{p}(r) = f(r)] \leq 1 - \Delta(f, F)$.

Examples of trivial tests:

The trivial tests for $LD[\mathbb{F}, n, tot \leq d]$ and $LD[\mathbb{F}, n, ind \leq d]$ are NOT optimal.

EXERCISE: where does the proof that the trivial test is optimal for $LD[\mathbb{F}, n=1, d]$ break down for $n > 1$?

F	ALL-ZERO	CONST	$LIN(\mathbb{F}, n)$	$LD[\mathbb{F}, n=1, d]$	$LD[\mathbb{F}, n, tot \leq d]$	$LD[\mathbb{F}, n, ind \leq d]$
$\lvert S \rvert$	0	1	n	$d+1$	$\binom{n+d}{d}$	$(d+1)^n$

(LECTURE 7 / prior slide)

The BLR test for $LIN(\mathbb{F}, n)$ has 3 queries, much better than the $(n+1)$-query trivial test.

The $(d+2)$-query trivial test for $LD[\mathbb{F}, n=1, d]$ is optimal.

The trivial tests for $LD[\mathbb{F}, n, tot \leq d]$ and $LD[\mathbb{F}, n, ind \leq d]$ have large query complexity.

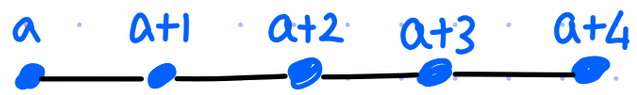
Today we see a total low-degree test that is much better than the trivial test.

Univariate Polynomials: a Different Attempt

We focus on a special case: $\mathbb{F} = \mathbb{F}_p$ for prime $p \geq d+2$.

The test is inspired by a different local characterization of low-degree polynomials:

<u>def:</u> For $i = 0, 1, \ldots, d+1$, $c_i := (-1)^{i+1}\binom{d+1}{i} \in \mathbb{F}_p$.



<u>lemma:</u> $\forall\, f: \mathbb{F}_p \to \mathbb{F}_p \quad \deg(f) \leq d \;\leftrightarrow\; \forall\, a \in \mathbb{F}_p \quad \sum_{i=0}^{d+1} c_i \cdot f(a+i) = 0$

The proof is by induction, using formal derivatives.

The derivative of $f(x)$ is $f'(x) := \frac{f(x+1)-f(x)}{(x+1)-x} = f(x+1) - f(x)$.

If $\deg(f) > 0$ then $\deg(f') = \deg(f) - 1$.

Ex for $d=0$: $(c_0, c_1) = (-1, 1) \to -f(a) + f(a+1) = 0$.

Ex for $d=1$: $(c_0, c_1, c_2) = (-1, 2, -1) \to -f(a) + 2f(a+1) - f(a+2) = 0$, i.e., $\frac{f(a+1)-f(a)}{(a+1)-a} - \frac{f(a+2)-f(a+1)}{(a+2)-(a+1)} = 0$.

New proposal: $V^{f:\mathbb{F}_p \to \mathbb{F}_p}(\mathbb{F}_p, d) :=$

1. Sample $r \leftarrow \mathbb{F}_p$
2. Query f at $r, r+1, \ldots, r+(d+1)$
3. Check that $\sum_{i=0}^{d+1} c_i \cdot f(r+i) = 0$

PROBLEM: it does not work. [Not all local characterizations do!]

Consider $f =$ 

, which has distance $\geq \frac{1}{2} - \frac{d}{|\mathbb{F}|}$ to $LD[\mathbb{F}, n=1, d]$.

This test rejects with probability only $\Theta(d/|\mathbb{F}|)$.

A Refined Local Characterization

def: For $i=0,1,\dots,d+1$, $c_i := (-1)^{i+1}\binom{d+1}{i} \in \mathbb{F}_p$.

lemma: $\forall\, f:\mathbb{F}_p\to\mathbb{F}_p \quad \deg(f)\le d \;\leftrightarrow\; \forall a\in\mathbb{F}_p \quad \sum_{i=0}^{d+1} c_i\cdot f(a+i)=0$

corollary: $\forall\, f:\mathbb{F}_p\to\mathbb{F}_p \quad \deg(f)\le d \;\leftrightarrow\; \forall a,b\in\mathbb{F}_p \quad \sum_{i=0}^{d+1} c_i\cdot f(a+i\cdot b)=0$

proof:

For the direction "$\leftarrow$" set $b=1$ and invoke the lemma.

For the direction "$\rightarrow$", fix $a,b\in\mathbb{F}_p$ and consider $g(x):=f(a+xb)$.

The degree of g is at most d. Hence, by the lemma,

$$\forall\, e\in\mathbb{F}_p \qquad 0=\sum_{i=0}^{d+1} c_i\cdot g(e+i)=\sum_{i=0}^{d+1} c_i\cdot f(a+(e+i)\cdot b)=\sum_{i=0}^{d+1} c_i\cdot f((a+eb)+i\cdot b)\,.$$

Set $e:=0$, and we get the condition for a,b. ■

The local constraints increased from $|\mathbb{F}_p|=p$ to $|\mathbb{F}_p|^2=p^2$.

The choice of b randomizes the "step size" and seems to rule out the counterexample.

Univariate Polynomials: the Rubinfeld–Sudan Test

Check one of the $|\mathbb{F}_p|^2 = p^2$ local constraints at random.

$V_{RS}^{f:\mathbb{F}_p\to\mathbb{F}_p}(\mathbb{F}_p, d) :=$ 1. Sample $r, s \leftarrow \mathbb{F}_p$
2. Query f at $r, r+s, \ldots, r+(d+1)\cdot s$
3. Check that $\sum_{i=0}^{d+1} c_i \cdot f(r+i\cdot s) = 0$

query complexity: $d+2 = O(d)$ (& non-adaptive)

Completeness: if $\deg(f) \le d$ then $\Pr[V_{RS}^f = 1] = 1$ by corollary

Soundness: theorem: $\Pr[V_{RS}^f = 0] \ge \min\left\{\Omega\left(\frac{1}{d^2}\right), \frac{1}{2}\cdot\Delta(f, LD[\mathbb{F}, n=1, d])\right\}$

Equivalently: $\Pr[V_{RS}^f = 1] \le \max\left\{1 - O\left(\frac{1}{d^2}\right), 1 - \frac{1}{2}\cdot\Delta(f, LD[\mathbb{F}, n=1, d])\right\}$

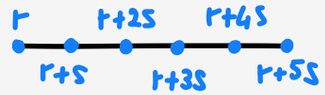
Isn't this test worse?

- lose a factor of 2 in distance (previously, $\Pr[V^f = 0] \ge \Delta(f, LD[\mathbb{F}, n=1, d])$)
- high-agreement regime: even if f is $\frac{1}{10}$-far, we get error only $\le 1 - O\left(\frac{1}{d^2}\right)$, so we need to repeat the test $O(d^2)$ times for constant error $\to O(d^3)$ queries

BUT: RS test extends to multivariate polynomials with no changes

Proof overview

$V_{RS}^{f:\mathbb{F}_p\to\mathbb{F}_p}$:= 1. Sample $r,s\in\mathbb{F}_p$



2. Check that $\sum_{i=0}^{d+1} c_i \cdot f(r+i\cdot s) = 0$

Similar to the case of linearity testing.

theorem: $\Pr\left[V_{RS}^f = 0\right] \geq \min\left\{\frac{1}{4\cdot(d+2)^2},\ \frac{1}{2}\cdot\Delta\left(f,\mathbb{F}_p^{\leq d}[x]\right)\right\}$.

The plurality correction is

$g_f:\mathbb{F}\to\mathbb{F}$ where $g_f(x) := \arg\max_{v\in\mathbb{F}_p}\left|\left\{s\in\mathbb{F}_p \mid v = \sum_{i=1}^{d+1} c_i\cdot f(x+i\cdot s)\right\}\right|$.

- Part 1: $\Pr\left[V_{RS}^f = 0\right] \geq \frac{1}{2}\cdot\Delta(f,g_f)$ far from plurality correction → many bad lines
- Part 2: $\Pr\left[V_{RS}^f = 0\right] < \frac{1}{4\cdot(d+2)^2} \to \deg(g_f)\leq d$ few bad lines → plurality correction is low-degree

Conclusion:

- If $\Pr\left[V_{RS}^f = 0\right] \geq \frac{1}{4\cdot(d+2)^2}$ then we are done.
- If $\Pr\left[V_{RS}^f = 0\right] < \frac{1}{4\cdot(d+2)^2}$ then (by Part 2) g_f is low-degree and (by Part 1) we get

$$\Pr\left[V_{RS}^f = 0\right] \geq \frac{1}{2}\cdot\Delta(f,g_f) \geq \frac{1}{2}\cdot\Delta\left(f,LD[\mathbb{F},n=1,d]\right).$$

■

Analysis of the RS Test - Part 1

$$\sum_{i=0}^{d+1} c_i \cdot f(r+is) = 0 \iff f(r) = \sum_{i=1}^{d+1} c_i \cdot f(r+is)$$

The plurality correction of f is $g_f(x) := \arg\max_{v \in \mathbb{F}_p} \left| \left\{ s \in \mathbb{F}_p \;\middle|\; v = \sum_{i=1}^{d+1} c_i \cdot f(x+is) \right\} \right|$.

If g_f is far from f then V_{RS}^f rejects with high probability:

<u>claim:</u> $\Pr\left[V_{RS}^f = 0\right] \geq \frac{1}{2} \cdot \Delta(f, g_f)$

<u>proof:</u> Define $S := \left\{ r \in \mathbb{F}_p \;\middle|\; \Pr_s\left[f(r) \neq \sum_{i=1}^{d+1} c_i \cdot f(r+i \cdot s) \right] \geq \frac{1}{2} \right\}$.

For every $r \notin S$, $\Pr_s\left[f(r) = \sum_{i=1}^{d+1} c_i \cdot f(r+is) \right] > \frac{1}{2}$ (more than half of s's vote for $f(r)$)

so $f(r) = g_f(r)$.

Hence $\Delta(f, g_f) \leq \frac{|S|}{|\mathbb{F}|}$ ($\forall r$ if $f(r) \neq g_f(r)$ then $r \in S$).

$$\begin{aligned}
\text{So } \Pr\left[V_{RS}^f = 0\right] &= \Pr_r[r \in S] \cdot \Pr_{r,s}\left[V_{RS}^f = 0 \mid r \in S \right] + \Pr_r[r \notin S] \cdot \Pr_{r,s}\left[V_{RS}^f = 0 \mid r \notin S \right] \\
&\geq \frac{|S|}{|\mathbb{F}|} \cdot \min_{r \in S} \left\{ \Pr_s\left[f(r) \neq \sum_{i=1}^{d+1} c_i \cdot f(r+is) \right] \right\} + 0 \\
&\geq \frac{|S|}{|\mathbb{F}|} \cdot \frac{1}{2} \geq \Delta(f, g_f) \cdot \frac{1}{2}.
\end{aligned}$$

■

Analysis of the RS Test - Collision Lemma

few bad lines imply many votes for the plurality correction

claim: $\forall r \in \mathbb{F}_p,\ \Pr_s\left[g_f(r) = \sum_{i=1}^{d+1} c_i \cdot f(r+i\cdot s) \right] \geq 1 - 2\cdot(d+1)\cdot \Pr\left[V_{RS}^f = 0\right]$

proof:

$$\Pr_s\left[g_f(r) = \sum_{i=1}^{d+1} c_i \cdot f(r+i\cdot s)\right] = \max_{v\in\mathbb{F}_p} \Pr_s\left[v = \sum_{i=1}^{d+1} c_i \cdot f(r+i\cdot s)\right]$$

$\sum_i p_i^2 \leq \max_i\{p_i\} \cdot \sum_i p_i$ →

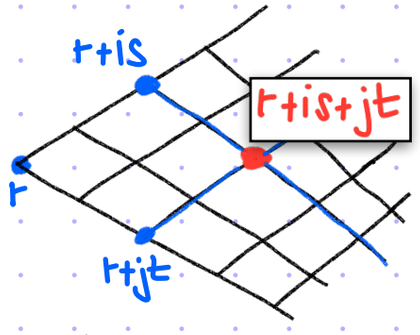
$$\geq \sum_{v\in\mathbb{F}_p} \Pr_s\left[v = \sum_{i=1}^{d+1} c_i \cdot f(r+i\cdot s)\right]^2$$

$$= \Pr_{s,t}\left[\sum_{i=1}^{d+1} c_i \cdot f(r+i\cdot s) = \sum_{i=1}^{d+1} c_i \cdot f(r+i\cdot t)\right]$$

$$\geq 1 - 2\cdot(d+1)\cdot \Pr\left[V_{RS}^f = 0\right].$$

We now analyze the COLLISION PROBABILITY.

For every $s,t \in \mathbb{F}$ if $\begin{cases} \forall i \in \{1,\ldots,d+1\} & f(r+is) = \sum_{j=1}^{d+1} c_j \cdot f((r+is)+jt) \\ \forall j \in \{1,\ldots,d+1\} & f(r+jt) = \sum_{i=1}^{d+1} c_i \cdot f((r+jt)+is) \end{cases}$



then $\sum_{i=1}^{d+1} c_i \cdot f(r+is) = \sum_{i=1}^{d+1} c_i \cdot \sum_{j=1}^{d+1} c_j \cdot f((r+is)+jt) = \sum_{j=1}^{d+1} c_j \sum_{i=1}^{d+1} c_i \cdot f((r+jt)+is) = \sum_{j=1}^{d+1} c_j \cdot f(r+jt)$.

Hence

$$\Pr_{s,t}\left[\sum_{i=1}^{d+1} c_i \cdot f(r+is) \neq \sum_{i=1}^{d+1} c_i \cdot f(r+it)\right] \leq \Pr_{s,t}\left[\begin{array}{l} \exists i \in \{1,\ldots,d+1\} \quad f(r+is) \neq \sum_{j=1}^{d+1} c_j \cdot f((r+is)+jt) \\ \text{or} \\ \exists j \in \{1,\ldots,d+1\} \quad f(r+jt) \neq \sum_{i=1}^{d+1} c_i \cdot f((r+jt)+is) \end{array}\right]$$

$$\leq 2(d+1)\cdot \Pr\left[V_{RS}^f = 0\right].$$

■

Analysis of the RS Test - Part 2

claim: if $\Pr\left[V_{RS}^{f}=0\right] < \frac{1}{4\cdot(d+2)^2}$ then $\deg(q_f) \leq d$

proof: Fix $r,s \in \mathbb{F}_p$. We show that $\sum_{i=0}^{d+1} c_i \cdot q_f(r+is) = 0$.

If $\exists\, t_1, t_2 \in \mathbb{F}_p$ s.t. $\begin{cases} \forall\, i \in \{0,1,\ldots,d+1\} \\ \forall\, j \in \{1,\ldots,d+1\} \end{cases}$ $\begin{array}{l} q_f(r+is) = \sum_{j=1}^{d+1} c_j \cdot f\left((r+is)+j\,(t_1+it_2)\right) \\ \sum_{i=0}^{d+1} c_i \cdot f\left((r+jt_1)+i\,(s+jt_2)\right) = 0 \end{array}$

then

$$\sum_{i=0}^{d+1} c_i \cdot q_f(r+is) = \sum_{i=0}^{d+1} c_i \cdot \left[\sum_{j=1}^{d+1} c_j \cdot f\left((r+is)+j\,(t_1+it_2)\right)\right] \overset{\text{reorder summations}}{=} \sum_{j=1}^{d+1} c_j \cdot \left[\sum_{i=0}^{d+1} c_i \cdot f\left((r+jt_1)+j\,(t_1+it_2)\right)\right] = \sum_{j=1}^{d+1} c_j \cdot 0 = 0.$$

Hence

$$\Pr_{t_1,t_2}\left[\sum_{i=0}^{d+1} c_i \cdot q_f(r+is) \neq 0\right] \overset{\text{union bound}}{\leq} \Pr_{t_1,t_2}\left[\begin{array}{ll} \exists\, i \in \{0,1,\ldots,d+1\} & q_f(r+is) \neq \sum_{j=1}^{d+1} c_j \cdot f\left((r+is)+j\,(t_1+it_2)\right) \\ \text{OR} & \\ \exists\, j \in \{1,\ldots,d+1\} & \sum_{i=0}^{d+1} c_i \cdot f\left((r+jt_1)+i\,(s+jt_2)\right) \neq 0 \end{array}\right]$$

$$\leq (d+2)\cdot\frac{1}{2\cdot(d+2)} + (d+1)\cdot\frac{1}{4\cdot(d+2)^2} < 1.$$

$$\Pr_{t_1,t_2}\left[q_f(r+is) \neq \sum_{j=1}^{d+1} c_j \cdot f\left((r+is)+j\cdot(t_1+it_2)\right)\right] \overset{\text{collision lemma}}{<} 2(d+1)\cdot\Pr\left[V_{RS}^{f}=0\right] < 2(d+1)\cdot\frac{1}{4\cdot(d+2)^2} \leq \frac{1}{2\cdot(d+2)}$$

■

Extending the RS Test to Multivariate Polynomials

The local characterization holds similarly:

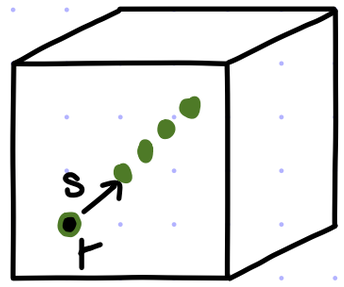
$$\forall\, d < p-2 \quad \forall\, f : \mathbb{F}_p^n \to \mathbb{F}_p, \quad \deg_{tot}(f) \le d \iff \forall\, a,b \in \mathbb{F}_p^n \ \sum_{i=0}^{d+1} c_i \cdot f(a+i\cdot b) = 0$$

This directly motivates the following test:

query complexity is $d+2 = O(d)$

$V_{RS}^{f:\mathbb{F}_p^n \to \mathbb{F}_p}(\mathbb{F}_p, n, d) :=$

1. Sample $r, s \leftarrow \mathbb{F}_p^n$
2. Query f at $r, r+s, \ldots, r+(d+1)\cdot s$
3. Check that $\sum_{i=0}^{d+1} c_i \cdot f(r+i\cdot s) = 0$



read $d+2$ locations on a random line

The theorem for soundness is also similar:

theorem: $\Pr\left[V_{RS}^{f} = 0\right] \ge \min\left\{\frac{1}{4\cdot(d+2)^2}, \frac{1}{2}\cdot\Delta\left(f, LD[\mathbb{F}_p, n, tot \le d]\right)\right\}$

The proof is the same up to syntactic modifications.

By repeating the test $O(d^2)$ times, we get:

a total low-degree test with query complexity $O(d^3)$ [independent of n]

where "constant relative distance" → "constant soundness error".

More on Total Low-Degree Testing [1/2]

A key structure that enables low-degree testing is a ROBUST LINES CHARACTERIZATION.

Suppose that $f:\mathbb{F}^n \to \mathbb{F}$ has total degree $\leq d$.

Then, $\forall a,b \in \mathbb{F}^n$, the univariate function $g_{a,b}(z) := f(a+z\cdot b)$ has degree $\leq d$.

Does the converse hold?

COUNTEREXAMPLE: Set $\mathbb{F} := \mathbb{F}_{p^e}$ and fix any d with $p^e - p^{e-1} - 1 < d < p^e$.

Consider the bivariate function $f(x_1,x_2) := (x_1^{p-1} x_2)^{p^{e-1}}$.

The total degree of f is $p^e > d$. Yet $\forall a,b$ $g_{a,b}(z) = f(a_1+zb_1, a_2+zb_2) = ((a_1+zb_1)^{p-1}(a_2+zb_2))^{p^{e-1}}$ has degree at most $(p-1)p^{e-1} = p^e - p^{e-1}$. (Indeed recall that $z^{p^e} \equiv z$ since p^e is the field size.)

The converse holds if $d \leq p^e - p^{e-1} - 1$. (E.g. if $\mathbb{F}$ has prime size p then the condition is $d \leq p-2$.) [Friedl, Sudan 1995] for a proof.

In this case, if $\{g_{a,b}(z) := f(a+zb)\}_{a,b\in\mathbb{F}^n}$ all have degree $\leq d$ then f has total degree $\leq d$.

Low-degree testing is based on distance variants of such results:

if $\{g_{a,b}(z) := f(a+zb)\}_{a,b\in\mathbb{F}^n}$ are close to degree $\leq d$ in expectation

then f is close to total degree $\leq d$

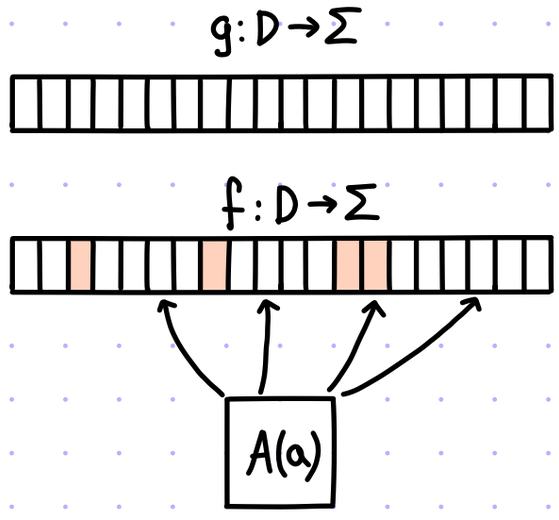
Local Correction of Low-Degree Functions [1/4]

Let $F = \{g : D \to \Sigma\}$ be a set of functions.

A local corrector for F with decoding error ε is an algorithm A s.t.

① $\forall g \in F \; \forall a \in D \quad \Pr[A^g(a) = g(a)] = 1$.

② $\forall g \in F \; \forall \delta \in [0,1] \; \forall f \text{ with } \Delta(f,g) \leq \delta \; \forall a \in D \quad \Pr[A^f(a) \neq g(a)] \leq \varepsilon(\delta)$.



The local corrector tolerates a δ^*-fraction of errors if $\varepsilon(\delta) \in [0, \frac{1}{2}) \; \forall \delta \in [0, \delta^*)$.

An error $< \frac{1}{2}$ enables error reduction via repetition and returning the plurality value.

Let $\Delta(F) := \min_{f,g \in F} \Delta(f,g)$ be the minimum relative distance of F.

claim: $\delta^* \leq \frac{1}{2}\Delta(F)$ (unique decoding radius)

proof: Fix any δ s.t. $\varepsilon(\delta) < \frac{1}{2}$. Fix arbitrary $g \in F$. There cannot be $g' \in F$ with $g' \neq g$ and $\Delta(g',g) \leq \delta$ because, picking $a \in D$ s.t. $g'(a) \neq g(a)$, cannot have $\Pr[A(a)^g = g(a)] > \frac{1}{2}$ and $\Pr[A(a)^g = g'(a)] > \frac{1}{2}$. Hence $\Delta(F) > 2\delta$.

We saw a 2-query local corrector for $\mathrm{LIN}[\mathbb{F}, n]$ with $\varepsilon(\delta) = 2\delta$, for which $\delta^* = \frac{1}{4}$.

We discuss several local correctors for $\mathrm{LD}[\mathbb{F}, n, \mathrm{tot} \leq d]$:

- A_1 makes $d+1$ queries and has error $\varepsilon(\delta) = (d+1)\cdot\delta$, so $\delta^* = \frac{1}{2(d+1)}$.
- A_2 makes $q-1$ queries and has error $\varepsilon(\delta) = \frac{2}{1-\frac{d}{q-1}} \cdot \delta$, so $\delta^* = \frac{1}{4} \cdot \left(1 - \frac{d}{q-1}\right)$.
- A_3 makes $q-1$ queries and has error $\varepsilon(\delta) = O_{d,\delta}\left(\frac{1}{q}\right)$ for $\delta < \frac{1}{2} \cdot \left(1 - \frac{2d}{q-1}\right)$, so $\delta^* = \frac{1}{2} \cdot \left(1 - \frac{2d}{q-1}\right)$.

We let $q := |\mathbb{F}|$ here and in next few slides.

Local Correction of Low-Degree Functions [2/4]

Suppose that $f:\mathbb{F}^n \to \mathbb{F}$ is δ-close to $g \in LD[\mathbb{F},n,tot\le d]$.

We wish to correct the value at the point $a \in \mathbb{F}^n$.

IDEA: focus on the values of f on the line $\ell_{a,b} := \{a+\lambda b\}_{b\in\mathbb{F}}$ for a random $b \in \mathbb{F}^n$.

Recall that a polynomial of degree d is determined by its evaluations at any $d+1$ points.

Let $\{\lambda_1,\dots,\lambda_{d+1}\} \subseteq \mathbb{F}\setminus\{0\}$ be a set of size $d+1$.

We attempt to recover $f(a)$ from $f(a+\lambda_1 b),\dots,f(a+\lambda_{d+1} b)$.

$A_1^f(a) :=$ 1. Sample $b \leftarrow \mathbb{F}^n$.
2. Query f at $\{a+\lambda_i b\}_{i=1}^{d+1}$.
3. Let $p(x)$ be the interpolation of $\{(a+\lambda_i b, f(a+\lambda_i b))\}_{i=1}^{d+1}$.
4. Output $p(0)$.

claim: $\forall a \in \mathbb{F}^n \ \Pr[A_1^f(a) \neq g(a)] \le (d+1)\delta$.

proof: For every $i \in [d+1]$, $\Pr_b[f(a+\lambda_i b) \neq g(a+\lambda_i b)] \le \delta$.

Hence $\Pr[p(0) \neq g(a)] = \Pr[\exists i \in [d+1] \ f(a+\lambda_i b) \neq g(a+\lambda_i b)] \le (d+1)\cdot\delta$. ■

Local Correction of Low-Degree Functions [3/4]

We see a local corrector that makes $q-1$ queries and tolerates more errors. (The special case $q=d+2$ is the previous local corrector.)

IDEA: the values along a line have high redundancy and it suffices to find a line with few enough errors.

$A_2^f(a)$:=
1. Sample $b \leftarrow \mathbb{F}^n$.
2. Query f at $\{a+\lambda b\}_{\lambda \in \mathbb{F}\setminus\{0\}}$.
3. Let $p(x)$ be the unique polynomial of degree d s.t. $|\{\lambda \in \mathbb{F}\setminus\{0\} : p(\lambda) \neq f(a+\lambda b)\}| < \frac{q-1-d}{2}$.
4. Abort if no $p(x)$ is found; else output $p(0)$.

← can efficiently find $p(x)$ via the Berlekamp–Welch algorithm

claim: $\forall a \in \mathbb{F}^n \ \Pr[A_2^f(a) \neq g(a)] \leq \frac{2\delta}{1-\frac{d}{q-1}}$.

More generally, can modify A_2 to obtain $p(x)$ by querying f at $\{a+\lambda b\}_{\lambda \in S}$ for any $S \subseteq \mathbb{F}\setminus\{0\}$ with $|S| \geq d+1$. In this case A_2 searches for p of degree d s.t. $|\{\lambda \in S : p(\lambda) \neq f(a+\lambda b)\}| < \frac{|S|-d}{2}$ and errs w.p. $\leq \frac{2\delta}{1-\frac{d}{|S|}}$.

proof: For every $\lambda \in \mathbb{F}\setminus\{0\}$, let Z_λ be the indicator for the event $f(a+\lambda b) \neq g(a+\lambda b)$, and note that $\mathbb{E}[Z_\lambda] = \Pr[Z_\lambda = 1] \leq \delta$. Define $Z := \sum_{\lambda \in \mathbb{F}\setminus\{0\}} Z_\lambda$ and note that $\mathbb{E}[Z] \leq (q-1)\delta$.

If $Z < \frac{q-1-d}{2}$ then $p(x) \equiv g(a+xb)$ so $p(0) = g(a)$ (correction succeeds).

Hence $\Pr[p(0) \neq g(a)] \leq \Pr[Z \geq \frac{q-1-d}{2}] \leq \frac{2}{q-1-d} \cdot \mathbb{E}[Z] \leq \frac{2}{q-1-d} \cdot (q-1) \cdot \delta = \frac{2\delta}{1-\frac{d}{q-1}}$. ■

↑ Markov's inequality: for a random variable $Z \geq 0$ and $c > 0$ $\Pr[Z \geq c] \leq \frac{1}{c}\mathbb{E}[Z]$.

Local Correction of Low-Degree Functions [4/4]

An additional improvement tolerates even more errors, close to the unique decoding radius $\frac{1}{2}\cdot\left(1-\frac{d}{q}\right)$.

The idea is to correct on a random quadratic curve, whose locations are pairwise independent.

$A_3^f(a) :=$
1. Sample $b, c \leftarrow \mathbb{F}^n$.
2. Query f at $\{a+\lambda b+\lambda^2 c\}_{\lambda\in\mathbb{F}\setminus\{0\}}$.
3. Let $p(x)$ be the unique polynomial of degree $2d$ s.t. $\left|\{\lambda\in\mathbb{F}\setminus\{0\} : p(\lambda)\neq f(a+\lambda b+\lambda^2 c)\}\right| < \frac{q-1-2d}{2}$.
4. Abort if no $p(x)$ is found; else output $p(0)$.

← can efficiently find $p(x)$ via the Berlekamp–Welch algorithm

claim: $\forall a\in\mathbb{F}^n \ \Pr\left[A_3^f(a)\neq g(a)\right] \leq \frac{1}{q-1}\cdot\frac{4\cdot(\delta-\delta^2)}{\left(1-\frac{2d}{q-1}-\delta\right)^2}$.

Can similarly modify A_3 to obtain $p(x)$ by querying f at $\{a+\lambda b+\lambda^2 c\}_{\lambda\in S}$ for any $S\subseteq\mathbb{F}\setminus\{0\}$ with $|S|\geq d+1$.

proof: For every $\lambda\in\mathbb{F}\setminus\{0\}$, let Z_λ be the indicator for the event $f(a+\lambda b+\lambda^2 c)\neq g(a+\lambda b+\lambda^2 c)$ and note that $\mathbb{E}[Z_\lambda]=\Pr[Z_\lambda=1]\leq\delta$ and $\mathrm{Var}(Z_\lambda)=\mathbb{E}[Z_\lambda]-\mathbb{E}[Z_\lambda]^2\leq\delta-\delta^2$ ($x-x^2$ is increasing on $[0,\frac{1}{2}]$).

Define $Z:=\sum_{\lambda\in\mathbb{F}\setminus\{0\}} Z_\lambda$ and note that $\mathbb{E}[Z]\leq(q-1)\delta$ and $\mathrm{Var}(Z)=\sum_{\lambda\in\mathbb{F}\setminus\{0\}}\mathrm{Var}(Z_\lambda)\leq(q-1)\cdot(\delta-\delta^2)$.
(↑ pairwise independence)

If $Z<\frac{q-1-2d}{2}$ then $p(x)\equiv g(a+xb+x^2c)$ so $p(0)=g(a)$ (correction succeeds).

Hence $\Pr[p(0)\neq g(a)]\leq\Pr\left[Z\geq\frac{q-1-2d}{2}\right]\leq\Pr\left[|Z-\mathbb{E}[X]|\geq\frac{q-1-2d}{2}-\mathbb{E}[X]\right]$

$\leq\Pr\left[|Z-\mathbb{E}[X]|\geq\frac{q-1-2d}{2}-(q-1)\delta\right]\leq\frac{\mathrm{Var}(Z)}{\left(\frac{q-1-2d}{2}-(q-1)\delta\right)^2}\leq\frac{(q-1)(\delta-\delta^2)}{\left(\frac{q-1-2d}{2}-(q-1)\delta\right)^2}=\frac{1}{q-1}\cdot\frac{4\cdot(\delta-\delta^2)}{\left(1-\frac{2d}{q-1}-\delta\right)^2}$. ■

(↑ Chebyshev's inequality: for a random variable Z and $c>0$ $\Pr[|Z-\mathbb{E}[Z]|\geq c]\leq\frac{1}{c^2}\cdot\mathrm{Var}(Z)$.)

A few words about low-degree testing

Low-degree testing for quantum states, and a quantum entangled games PCP for QMA

Anand Natarajan* Thomas Vidick†

Abstract

We show that given an explicit description of a multiplayer game, with a classical verifier and a constant number of players, it is QMA-hard, under randomized reductions, to distinguish between the cases when the players have a strategy using entanglement that succeeds with probability 1 in the game, or when no such strategy succeeds with probability larger than $\frac{1}{2}$. This proves the "games quantum PCP conjecture" of Fitzsimons and the second author (ITCS'15), albeit under randomized reductions.

The core component in our reduction is a construction of a family of two-player games for testing n-qubit maximally entangled states. For any integer $n \geq 2$, we give such a game in which questions from the verifier are $O(\log n)$ bits long, and answers are $\mathrm{poly}(\log \log n)$ bits long. We show that for any constant $\varepsilon \geq 0$, any strategy that succeeds with probability at least $1 - \varepsilon$ in the test must use a state that is within distance $\delta(\varepsilon) = O(\varepsilon^c)$ from a state that is locally equivalent to a maximally entangled state on n qubits, for some universal constant $c > 0$. The construction is based on the classical plane-vs-point test for multivariate low-degree polynomials of Raz and Safra (STOC'97). We extend the classical test to the quantum regime by executing independent copies of the test in the generalized Pauli X and Z bases

Low-degree tests at large distances

Alex Samorodnitsky*

September 27, 2018

Abstract

We define tests of boolean functions which distinguish between linear (or quadratic) ...te sense, from these poly- ...etween soundness and the ...ormity norms behave "ran- ...f of an inverse theorem for

Testing Low-Degree Polynomials over $GF(2)$

Noga Alon * Tali Kaufman † Michael Krivelevich ‡ Simon Litsyn § Dana Ron¶

July 9, 2003

Abstract

We describe an efficient randomized algorithm to test if a given binary function $f : \{0,1\}^n \to \{0,1\}$ is a low degree polynomial (that is, a sum of low degree monomials). For a given integer $k \geq 1$ and a given real $\epsilon > 0$, the algorithm queries f at $O(\frac{1}{\epsilon} + k4^k)$ points. If f is a polynomial of degree at most k, the algorithm always accepts, and if the value of f has to be modified on at least an ϵ fraction of all inputs in order to transform it to such a polynomial, then the algorithm rejects with probability at least 2/3. Our result is essentially tight: Any algorithm for testing degree-k polynomials over $GF(2)$ must perform $\Omega(\frac{1}{\epsilon} + 2^k)$ queries.

Improved low-degree testing and its applications

Sanjeev Arora* Princeton University

Madhu Sudan† IBM T. J. Watson Research Center

Abstract

NP = PCP(log n, 1) and related results crucially depend upon the close connection between the probability with which a function passes a *low degree test* and the distance of this function to the nearest degree d polynomial. In this paper we study a test proposed by Rubinfeld and Sudan [29]. The strongest previously known connection for this test states that a function passes the test with probability δ for some $\delta > 7/8$ iff the function has agreement $\approx \delta$ with a

1 Introduction

The use of algebraic techniques h... (probabilistic) characterizations of t... classes. These characterizations inve... tween an untrustworthy prover (or ... polynomial-time verifier. In MIP= ... NP = PCP(log n, 1) [6, 5] the verifi... cally verify the satisfiability of a boo... ing very few bits in a "proof string" ... In IP=PSPACE [24, 31] the verifier h...

A Sub-Constant Error-Probability Low-Degree Test, and a Sub-Constant Error-Probability PCP Characterization of NP *

Ran Raz † Shmuel Safra ‡

Abstract

We introduce a new low-degree test, one that uses the restriction of low-degree polynomials to planes (i.e., affine sub-spaces of dimension 2), rather than the restriction to lines (i.e., affine sub-spaces of dimension 1). We prove the new test to be of a very small error-probability (in particular, much smaller than constant).

The new test enables us to prove a low-error characterization of NP in terms of PCP. Specifically, our theorem states that, for any given $\epsilon > 0$, membership in any NP language can be verified with $O(1)$ accesses, ...

of the most fundamental avenues of research in theory of computer-science.

Since the early days, when the classes P and NP were defined, and the question was posed as to whether they are the same or do they differ, many problems were shown to be NP-complete, thereby increasing the weight on finding stricter characterization for the class NP.

NP has since been given a few alternative characterizations. The one most commonly applied being Cook's [Coo71], which characterizes NP in terms of efficient verification of proofs (or nondeterministic computations).

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